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反对称铺设复合材料层合板非线性后屈曲分析

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摘　要:基于层合板壳理论,考虑反对称铺设层合板的拉弯耦合效应和后屈曲过程中的非线性几何变形,推导了由应力函数和挠度表示的复合材料层合板的后屈曲控制方程。引入无量纲参数对控制方程和边界条件进行无量纲化,以消除材料参数及几何尺寸对分析结果的影响。采用摄动法将无量纲的非线性控制方程及边界条件展开成一系列非齐次线性摄动方程组,分析各阶摄动方程的通解与特解的构造,并逐次求解,建立了反对称铺设复合材料层合板受单向均布压力作用的临界屈曲荷载及后屈曲平衡路径的理论解。进而运用 ABAQUS 软件对复合材料层合板在面内压缩载荷作用下的屈曲和后屈曲进行有限元分析,结果表明理论解与 ABAQUS 结果十分接近,验证了理论解的正确性。在此基础上进一步讨论了铺设角度、铺设层数和拉弯耦合效应等对层合板后屈曲性能的影响。研究发现层合板的屈曲载荷受铺设角度与层数的影响较为显著,而拉弯耦合效应使板的屈后强度大大降低。

关键词:复合材料层合板;后屈曲;摄动法;有限元分析

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Nonlinear post-buckling analysis of composite laminates with antisymmetric angle-ply

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Abstract: Based on the theory for laminated plates and shells, governing equations for post-buckling of anti-symmetrical laminated composite plates are derived in terms of stress function and transverse deflection by considering the tension-bending coupling effect and the von-Kaman nonlinear geometric relation. By introducing the non-dimensional parameters, the nonlinear governing equations and the boundary conditions are nondimensionalized to eliminate the influence of the material properties and the geometric parameters. Perturbation approach is applied to expand the governing equations as well as the boundary conditions into a series equations of a small perturbation parameter. Analytical solutions of the critical buckling load and the post-buckling equilibrium path for anti-symmetric cross plied composite laminates with simply supported

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boundary conditions are obtained by examining the characteristics of the general and particular solutions. After that, the buckling and post-buckling of the composite laminates with antisymmetric angle-ply under in-plane compressive load are analyzed using ABAQUS software, and the results from ABAQUS are compared with the theoretical solutions. It shows that the two results are in very good agreement which verifies the accuracy and effectiveness of the proposed analytical approach. Using the established solutions, the influences of the orientation angle of the composite ply, the number of the composite ply and tension-bending coupling effect on the post-buckling performance of the laminate are discussed. It is found that the critical buckling load of the laminate plate is significantly affected by the orientation angle and the tension-bending coupling effect will greatly reduce the post-buckling load carry capability of the laminates.

Key words: composite laminate; post-buckling; perturbation approach; finite element analysis

复合材料层合板壳结构,因为轻量化、高比强度和比模量等优点,通常作为承载部件广泛的应用于机械化工、航空航天等领域。承受面内压缩荷载作用的板壳结构,当承载力增加到某一极限载荷值时,结构很容易会发生失稳破坏,进而给工业和日常使用带来极大危害,因此层合结构的稳定性分析极具工程价值。

20 世纪初 Von Kármán 建立了著名的板壳非线性弯曲理论[1],奠定了板壳几何非线性分析的基础,从此世界各国学者们对复合材料层合板壳的非线性力学性能开始进行深入研究[2-3]。吴晓等[4]考虑扁壳大挠度弯曲时引起的中面应变,视双模量扁壳为两种材料组成的层合扁壳,解决了双模量简支扁壳在均匀内压作用下的非线性弯曲问题。BELLIFA 等[5]根据 Eringen 的非局部弹性理论,考虑剪切力的轴向位移效应,提出了一种针对纳米尺度梁的非线性后屈曲行为的非局域零阶剪切变形理论。张作亮等[6]利用 Von Kármán 大挠度的理论,结合达朗贝尔原理和薄板振动理论,建立了伞形张拉膜结构在冰雹荷载作用下的非线性动力响应控制方程。当板壳结构处于小变形和适度旋转的情况下时,如初始后屈曲阶段,采用 Von Kármán 非线性几何关系可以获得有效的结果,但当板壳结构处于深后屈曲阶段时,则必须考虑板壳结构的大变形。SHARIYAT 和 ASEMI[7]利用全格林应变张量来描述板壳结构深后屈曲过程中的几何非线性关系,并分析了受单轴和双轴压缩载荷作用下变刚度矩形板的后屈曲行为。

由于结构非线性弯曲问题的控制方程难以获得精确解,学术界常常采用数值法,如有限元法、渐进损伤分析法,来进行分析。ALMITANI[8]采用有限元分析方法研究了对称与反对称功能梯度梁的屈曲问题。杨刚等[9]提出了改进层合板壳的二维分层 Mindlin 模型,同样利用有限元法分析了含有圆形和椭圆形分层的复合材料层合板后屈曲行为。卢鑫瑞等[10]用有限元法研究了关于无人机 3 个关键部位的复合材料蒙皮壁加筋板的非线性屈曲问题,得到了壁板的屈曲特性和后屈曲损伤演变过程。林国伟等[11]和孙中雷等[12]均采用了渐进损伤分析方法,分别对复合材料加筋板在承受轴压载荷下的屈曲载荷及其破坏模式和后屈曲极限破坏载荷及其失效模式进行了深入的研究,前者改进了已有的工程计算方法,使计算结果与实验值更加吻合[11],后者分析了各种参数对复合材料加筋板后屈曲特性的影响[12],为复合材料加筋板的设计提供了参考依据。

在分析结构承受复杂外力作用下的力学性能时,有限元法具有不可替代的优势,而近似法,如伽辽金法[13-15]、Koiter 渐进法[16]、摄动法等,则能更深入解释结构失效的机制,揭示结构内部的受力情况。摄动法是当前解决工程技术问题和科学问题的主要数学工具之一,因此被广泛的用于求解结构的非线性问题。佘桂林[17]基于 Euler-Bernoulli 梁理论研究了热膨胀下梁的横向非线性振动问题,并采用二次摄动法进行求解,与传统摄动法的计算结果进行了对比验证。李萍等[18]采用小参数摄动法研究了脱层对复合材料层合梁的后屈曲的影响。YANGA J 等[19]采用摄动法讨论了功能梯度矩形板的大变形问题,再基于四边固支的边界条件,讨论了在预加横向载荷和面内压缩载荷作用下的后屈曲行为。DASH 等[20]基于拉格朗日非线性应变-位移关系的高阶剪切变形理论,利用摄动法研究了层压复合板的屈曲和后屈曲问题。夏飞等[21]同样运用摄动法研究了在横向剪切与湿热环境下温度、脱层条件等各参数对复合材料层合板的屈曲与后屈曲的影响。

由于正交对称铺设的层合板不存在拉弯耦合效

应,因此,上述文献在使用摄动法时得到的非齐次线性摄动方程组,可以通过对应力函数与挠度进行解耦来求解。而当铺设情况为反对称铺设时,拉弯耦合效应的存在导致应力函数与挠度无法解耦。本研究在轴压作用下的简支反对称铺设复合材料矩形层合板的后屈曲问题,将 Von Kármán 非线性几何方程引入到层合板的平衡微分方程中,建立了反对称铺设层合板的非线性后屈曲控制方程。通过使用二阶小参数摄动法,求解非线性控制方程组,得到了临界屈曲载荷与后屈曲平衡路径。应用 ABAQUS 软件进行仿真分析,并将有限元解与理论解进行比较,验证理论解的准确性与可靠性。最后进行算例分析,讨论不同铺设方式、拉弯耦合效应等对层合板的临界屈曲载荷及后屈曲性能的影响。

1 基本方程

1.1 模型的建立

考虑图 1 所示的四边简支的复合材料层合板,长度为 a,宽度为 b,铺层数为 N,总厚为 t,单层板厚为 h,受 x 方向均布荷载 N_x 作用,建立直角坐标系。各子板的几何尺寸如图 1 所示。

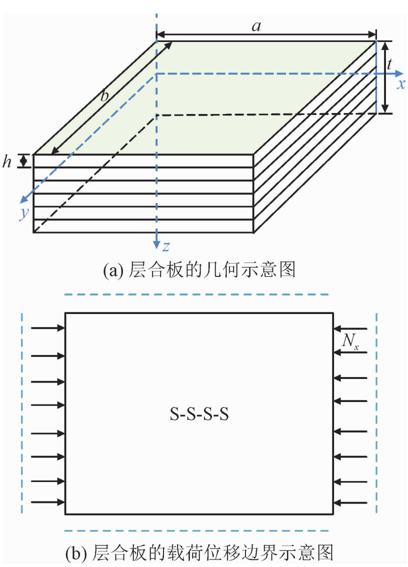

图 1 层合板模型示意图

Fig. 1 Model of rectangular plate

1.2 层合板的非线性大挠度方程

设 ε_x^0、ε_y^0、ε_{xy}^0 为层合板的中面应变,κ_x、κ_y、κ_{xy} 为中面的曲率、扭曲率,u、v、w 为该区域的中面位移,则由 Von Kármán 非线性板壳理论可得

$$\begin{bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \varepsilon_{xy}^0\end{bmatrix}=\begin{bmatrix}u_{,x}+(v_{,x})^2/2\\ v_{,y}+(w_{,y})^2/2\\ u_{,y}+v_{,x}+w_{,x}w_{,y}\end{bmatrix}\tag{1}$$

$$\begin{bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{bmatrix}=\begin{bmatrix}-w_{,xx}\\ -w_{,yy}\\ -2w_{,xy}\end{bmatrix}\tag{2}$$

式中,$u_{,x}$表示变量 u 对变量 x 求偏导,以此类推。从式(1)中消去 u 和 v,得到层合板非线性弯曲的协调方程为

$$\varepsilon_{x,yy}+\varepsilon_{y,xx}-\varepsilon_{xy,xy}=w_{,xy}^2-w_{,xx}w_{,yy}\tag{3}$$

本研究反对称铺设各向异性层合板,设 θ 为第 k 层单层板的坐标系与材料主方向的夹角,则该层的弹性刚度为

$$\begin{bmatrix}\bar{Q}_{11}^k\\ \bar{Q}_{12}^k\\ \bar{Q}_{22}^k\\ \bar{Q}_{16}^k\\ \bar{Q}_{26}^k\\ \bar{Q}_{66}^k\end{bmatrix}=\begin{bmatrix}C^4 & 2C^2S^2 & S^4 & 4C^2S^2\\ C^2S^2 & C^4+S^4 & C^2S^2 & -4C^2S^2\\ S^4 & 2C^2S^2 & C^4 & 4C^2S^2\\ C^3S & CS^3-C^3S & -CS^3 & -2CS(C^2-S^2)\\ CS^3 & C^3S-CS^3 & -C^3S & 2CS(C^2-S^2)\\ C^2S^2 & -2C^2S^2 & C^2S^2 & (C^2-S^2)^2\end{bmatrix}\begin{bmatrix}Q_{11}\\ Q_{12}\\ Q_{22}\\ Q_{66}\end{bmatrix}\tag{4}$$

式中:$C=\cos\theta$;$S=\sin\theta$,且

$$\begin{cases}Q_{11}=\dfrac{E_1}{1-\mu_{12}\mu_{21}},Q_{22}=\dfrac{E_2}{1-\mu_{12}\mu_{21}},Q_{66}=G_{12}\\ Q_{12}=\mu_{12}Q_{22}=\mu_{21}Q_{11},\mu_{12}E_2=\mu_{21}E_1\end{cases}\tag{5}$$

式中:E_1、E_2表示材料在 1、2 弹性主方向上的弹性模量;G_{12}表示材料在 1-2 平面内的剪切弹性模量;μ_{12}和 μ_{21}为不同方向的泊松比。

根据复合材料力学理论[22],层合板的本构方程为

$$\begin{bmatrix}N\\ M\end{bmatrix}=\begin{bmatrix}A_{ij} & B_{ij}\\ B_{ij} & D_{ij}\end{bmatrix}\begin{bmatrix}\varepsilon^0\\ \kappa\end{bmatrix}\tag{6}$$

式中:N 为层合板的中面薄膜内力;M 为层合板的内力矩;A_{ij}为拉伸刚度;B_{ij}为耦合刚度;D_{ij}为弯曲刚度,分别表示为

$$\begin{cases} A_{ij} = \sum_{k=1}^{n} (\bar{Q}_{ij})_k (z_k - z_{k-1}) \\ B_{ij} = \frac{1}{2}\sum_{k=1}^{n} (\bar{Q}_{ij})_k (z_k^2 - z_{k-1}^2) \\ D_{ij} = \frac{1}{3}\sum_{k=1}^{n} (\bar{Q}_{ij})_k (z_k^3 - z_{k-1}^3) \end{cases} \tag{7}$$

式中，z_k表示第 k 层单层板。根据经典板壳理论，层合板后屈曲的平衡方程为

$$\begin{cases} N_{x,x} + N_{xy,y} = 0 \\ N_{xy,x} + N_{y,y} = 0 \\ M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} = 0 \end{cases} \tag{8}$$

引入应力函数

$$N_x = \Phi_{,yy}, \quad N_y = \Phi_{,xx}, \quad N_{xy} = -\Phi_{,xy} \tag{9}$$

将式(6)和(9)代入到协调方程(3)和平衡方程(8)的第 3 式中，同时考虑到对反对称铺设层合板，有 $B_{11} = B_{22} = B_{12} = B_{66} = 0$

得到反对称铺设层合板非线性后屈曲方程为

$$\tilde{D}_{11} w_{,xxxx} + \tilde{D}_{22} w_{,yyyy} + (2\tilde{D}_{12} + 4\tilde{D}_{66}) w_{,xxyy} - (2\tilde{B}_{62} - \tilde{B}_{16})\Phi_{,xxxy} - (2\tilde{B}_{61} - \tilde{B}_{26})\Phi_{,xyyy} - \Phi_{,yy} w_{,xx} - \Phi_{,xx} w_{,yy} + 2\Phi_{,xy} w_{,xy} = 0 \tag{10}$$

$$A_{22}^{-1}\Phi_{,xxxx} + A_{11}^{-1}\Phi_{,yyyy} + (2A_{12}^{-1} + A_{66}^{-1})\Phi_{,xxyy} + (2\tilde{\tilde{B}}_{16} - \tilde{\tilde{B}}_{62}) w_{,xyyy} + (2\tilde{\tilde{B}}_{26} - \tilde{\tilde{B}}_{61}) w_{,xxxy} = w_{,xy}^2 - w_{,xx} w_{,yy} \tag{11}$$

式中，$\tilde{B} = BA^{-1}$，$\tilde{\tilde{B}} = A^{-1}B$，$\tilde{D} = -BA^{-1}B + D$。

本研究对象为四边简支的层合板，边界条件可表示为

$$x = 0,\ a: \begin{cases} w = 0 \\ M_x = \tilde{B}_{61}\Phi_{,xy} + D_{11} w_{,xx} + \tilde{D}_{12} w_{,xy} = 0 \end{cases} \tag{12}$$

$$y = 0,\ b: \begin{cases} w = 0 \\ M_y = \tilde{B}_{26}\Phi_{,xy} + D_{12} w_{,xx} + \tilde{D}_{22} w_{,xy} = 0 \end{cases} \tag{13}$$

2　摄动法求解

本研究采用摄动法对式(10)和(11)进行求解。引入无量纲参数，即

$$\bar{x} = x/a,\ \bar{y} = y/b,\ \alpha = t/a,\ \beta = t/b,\ \bar{w} = w/t,$$
$$\bar{D}_{ij} = \tilde{D}_{ij}/A_{22}t^2,\ \bar{A}_{ij}^{-1} = A_{ij}^{-1}/A_{22},\ \bar{B}_{ij} = \bar{B}_{ij} = \tilde{B}_{ij}/t,$$
$$\bar{\bar{B}}_{ij} = \tilde{\tilde{B}}_{ij}/t,\ \bar{\Phi} = \Phi/A_{22}t^2,\ \bar{N}_x = N_x b^2/A_{22}t^2 \tag{14}$$

无量纲函数$\bar{w}$，$\bar{\Phi}$，$\bar{N}_x$ 展开式为

$$\bar{w} = \bar{w}_1 w_a w_0 + \bar{w}_2 w_a^2 w_0^2 + \bar{w}_3 w_a^3 w_0^3 + \cdots + \bar{w}_n w_a^n w_0^n,$$
$$\bar{\Phi} = \bar{\Phi}_0 + \bar{\Phi}_1 w_a w_0 + \bar{\Phi}_2 w_a^2 w_0^2 + \bar{\Phi}_3 w_a^3 w_0^3 + \cdots + \bar{\Phi}_n w_a^n w_0^n,$$
$$\bar{N}_x = \bar{N}_{\mathrm{cr}} + \bar{N}_{2h} w_a^2 w_0^2 + \bar{N}_{4h} w_a^4 w_0^4 + \cdots + \bar{N}_{2nh} w_a^{2n} w_0^{2n} \tag{15}$$

式中：$w_0 = (w)_{\max}/t$ 为摄动小参数；$\bar{N}_{\mathrm{cr}}$是反对称铺设复合材料层合板量纲为 1 的屈曲荷载。

将上述无量纲函数展开式代入控制方程，再对小参数 w_0的不同次幂进行合并，得到各阶摄动方程组，代入边界条件，逐阶求解便可获得足够精确的解。附录 A 给出了高阶求解结果。

零阶方程 $O(w_0^0)$ 为

$$(2\bar{B}_{62} - \bar{B}_{16})\alpha^3\beta\bar{\Phi}_{0,xxxy} + (2\bar{B}_{61} - \bar{B}_{26})\alpha\beta^3\bar{\Phi}_{0,xyyy} = 0,$$
$$\bar{A}_{22}^{-1}\alpha^4\bar{\Phi}_{0,xxxx} + \bar{A}_{11}^{-1}\beta^4\bar{\Phi}_{0,yyyy} + (2\bar{A}_{12}^{-1} + \bar{A}_{66}^{-1})\alpha^2\beta^2\bar{\Phi}_{0,xxyy} = 0 \tag{16}$$

满足边界均压应力的解为

$$\bar{\Phi}_0 = -\bar{N}_{\mathrm{cr}}\bar{y}^2/2 \tag{17}$$

一阶方程 $O(w_0^{'})$为

$$\bar{D}_{11}\alpha^4\bar{w}_{1,xxxx} + \bar{D}_{22}\beta^4\bar{w}_{1,yyyy} + (2\bar{D}_{12} + 4\bar{D}_{66})\cdot\alpha^2\beta^2\bar{w}_{1,xxyy} - (2\bar{B}_{62} - \bar{B}_{16})\alpha^3\beta\bar{\Phi}_{1,xxxy} - (2\bar{B}_{61} - \bar{B}_{26})\alpha\beta^3\bar{\Phi}_{1,xyyy} - \alpha^2\beta^2(\bar{\Phi}_{0,yy}\bar{w}_{1,xx} + \bar{\Phi}_{0,xx}\bar{w}_{1,yy}) + 2\alpha^2\beta^2\bar{\Phi}_{0,xy}\bar{w}_{1,xy} = 0,$$
$$\bar{A}_{22}^{-1}\alpha^4\bar{\Phi}_{1,xxxx} + \bar{A}_{11}^{-1}\beta^4\bar{\Phi}_{1,yyyy} + (2\bar{A}_{12}^{-1} + \bar{A}_{66}^{-1})\alpha^2\beta^2\bar{\Phi}_{1,xxyy} + (2\bar{\bar{B}}_{16} - \bar{\bar{B}}_{62})\alpha\beta^3\bar{w}_{1,xyyy} + (2\bar{\bar{B}}_{26} - \bar{\bar{B}}_{61})\alpha^3\beta\bar{w}_{1,xxxy} = 0 \tag{18}$$

满足边界条件的解为

$$\bar{w}_1 = w_a\sin(m\pi\bar{x})\sin(n\pi\bar{y})$$
$$\bar{\Phi}_1 = \varphi_a w_a\cos(m\pi\bar{x})\cos(n\pi\bar{y}) \tag{19}$$

其中

$$\varphi_a = \frac{(2\bar{\bar{B}}_{16} - \bar{\bar{B}}_{62})\alpha\beta^3 mn^3 + (2\bar{\bar{B}}_{26} - \bar{\bar{B}}_{61})\alpha^3\beta m^3 n}{\bar{A}_{22}^{-1}\alpha^4 m^4 + \bar{A}_{11}^{-1}\beta^4 n^4 (2\bar{A}_{12}^{-1} + \bar{A}_{66}^{-1})\alpha^2\beta^2 m^2 n^2} \tag{20}$$

$$\bar{N}_{\mathrm{cr}} = \pi^2\left[\frac{\bar{D}_{11}\alpha^2 m^2}{\beta^2} + \frac{\bar{D}_{22}\beta^2 n^4}{\alpha^2 m^2} + (2\bar{D}_{12} + 4\bar{D}_{66})n^2\right] + \pi^2\left[(2\bar{B}_{62} - \bar{B}_{16})\frac{\alpha mn}{\beta} + (2\bar{B}_{61} - \bar{B}_{26})\frac{\beta n^3}{\alpha m}\right]\varphi_a \tag{21}$$

二阶方程 $O(w_0^2)$ 为

$$\begin{aligned}&\overline{D}_{11}\alpha^4\overline{w}_{2,xxxx}+\overline{D}_{22}\beta^4\overline{w}_{2,yyyy}+(2\overline{D}_{12}+4\overline{D}_{66})\alpha^2\beta^2\overline{w}_{2,xxyy}-\\&(2\overline{B}_{62}-\overline{B}_{16})\alpha^3\beta\overline{\Phi}_{2,xxxy}-(2\overline{B}_{61}-\overline{B}_{26})\alpha\beta^3\overline{\Phi}_{2,xyyy}-\\&\alpha^2\beta^2(\overline{\Phi}_{0,yy}\overline{w}_{2,xx}+\overline{\Phi}_{1,yy}\overline{w}_{1,xx}+\overline{\Phi}_{1,xx}\overline{w}_{1,yy})+\\&2\alpha^2\beta^2\overline{\Phi}_{1,xy}\overline{w}_{1,xy}=0,\\&\overline{A}_{22}^{-1}\alpha^4\overline{\Phi}_{2,xxxx}+\overline{A}_{11}^{-1}\beta^4\overline{\Phi}_{2,yyyy}+(2\overline{A}_{12}^{-1}+\overline{A}_{66}^{-1})\alpha^2\beta^2\overline{\Phi}_{2,xxyy}+\\&(2\,\overline{\overline{B}}_{16}-\overline{\overline{B}}_{62})\alpha\beta^3\overline{w}_{2,xyyy}+(2\,\overline{\overline{B}}_{26}-\overline{\overline{B}}_{61})\alpha^3\beta\overline{w}_{2,xxxy}=\\&\alpha^2\beta^2(\overline{w}_{1,xy}^{\,2}-\overline{w}_{1,xx}\overline{w}_{1,yy})\end{aligned}\tag{22}$$

满足边界条件的解为

$$\begin{aligned}&\overline{w}_2=0,\\&\overline{\Phi}_2=\varphi_{21}w_a^2\cos(2m\pi\overline{x})+\varphi_{22}w_a^2\cos(2n\pi\overline{y})-\\&\quad\overline{N}_{2h}w_a^2\overline{y}^{\,2}/2\end{aligned}\tag{23}$$

其中

$$\begin{aligned}&\varphi_{21}=\beta^2n^2/32\overline{A}_{22}^{-1}\alpha^2m^2,\\&\varphi_{22}=\alpha^2m^2/32\overline{A}_{11}^{-1}\beta^2n^2\end{aligned}\tag{24}$$

三阶方程 $O(w_0^3)$ 为

$$\begin{aligned}&\overline{D}_{11}\alpha^4\overline{w}_{3,xxxx}+\overline{D}_{22}\beta^4\overline{w}_{3,yyyy}+(2\overline{D}_{12}+4\overline{D}_{66})\alpha^2\beta^2\overline{w}_{3,xxyy}-\\&(2\overline{B}_{62}-\overline{B}_{16})\alpha^3\beta\,\overline{\Phi}_{3,xxxy}-(2\overline{B}_{61}-\overline{B}_{26})\alpha\beta^3\,\overline{\Phi}_{3,xyyy}-\\&\alpha^2\beta^2(\overline{\Phi}_{0,yy}\overline{w}_{3,xx}+\overline{\Phi}_{2,yy}\overline{w}_{1,xx}+\overline{\Phi}_{2,xx}\overline{w}_{1,yy})=0,\\&\overline{A}_{22}^{-1}\alpha^4\overline{\Phi}_{3,xxxx}+\overline{A}_{11}^{-1}\beta^4\overline{\Phi}_{3,yyyy}+(2\overline{A}_{12}^{-1}+\\&\overline{A}_{66}^{-1})\alpha^2\beta^2\overline{\Phi}_{3,xxyy}+(2\,\overline{\overline{B}}_{16}-\overline{\overline{B}}_{62})\alpha\beta^3\overline{w}_{3,xyyy}+(2\,\overline{\overline{B}}_{26}-\\&\overline{\overline{B}}_{61})\alpha^3\overline{\beta w}_{3,xxxy}=0\end{aligned}\tag{25}$$

满足边界条件的解为

$$\begin{aligned}&\overline{w}_3=w_{31}w_a^3\sin(3m\pi\overline{x})\sin(n\pi\overline{y})+\\&\quad w_{32}w_a^3\sin(m\pi\overline{x})\sin(3n\pi\overline{y}),\\&\overline{\Phi}_3=\varphi_{31}w_a^3\cos(3m\pi\overline{x})\cos(n\pi\overline{y})+\\&\quad\varphi_{32}w_a^3\cos(m\pi\overline{x})\cos(3n\pi\overline{y})\end{aligned}\tag{26}$$

同理,四阶摄动方程 $O(w_0^4)$ 为

$$\begin{aligned}&\overline{D}_{11}\alpha^4\overline{w}_{4,xxxx}+\overline{D}_{22}\beta^4\overline{w}_{4,yyyy}+(2\overline{D}_{12}+4\overline{D}_{66})\alpha^2\beta^2\overline{w}_{4,xxyy}-\\&(2\overline{B}_{62}-\overline{B}_{16})\alpha^3\overline{\beta\Phi}_{4,xxxy}-(2\overline{B}_{61}-\overline{B}_{26})\alpha\beta^3\overline{\Phi}_{4,xyyy}-\\&\alpha^2\beta^2(\overline{\Phi}_{0,yy}\overline{w}_{4,xx}+\overline{\Phi}_{1,yy}\overline{w}_{3,xx}+\overline{\Phi}_{3,yy}\overline{w}_{1,xx}+\overline{\Phi}_{1,xx}\overline{w}_{3,yy}+\\&2\alpha^2\beta^2(\overline{\Phi}_{1,xy}\overline{w}_{3,xy}+\overline{\Phi}_{3,xy}\overline{w}_{1,xy})=0,\\&\overline{A}_{22}^{-1}\alpha^4\overline{\Phi}_{4,xxxx}+\overline{A}_{11}^{-1}\beta^4\overline{\Phi}_{4,yyyy}+(2\overline{A}_{12}^{-1}+\overline{A}_{66}^{-1})\alpha^2\beta^2\overline{\Phi}_{4,xxyy}+\\&(2\,\overline{\overline{B}}_{16}-\overline{\overline{B}}_{62})\alpha\beta^3\overline{w}_{4,xyyy}+(2\,\overline{\overline{B}}_{26}-\overline{\overline{B}}_{61})\alpha^3\beta\,\overline{w}_{4,xxxy}=\\&\alpha^2\beta^2(-\overline{w}_{1,xx}\overline{w}_{3,yy}-\overline{w}_{3,xx}\overline{w}_{1,yy})\end{aligned}\tag{27}$$

满足边界条件的解为

$$\begin{aligned}\overline{w}_4=\;&w_{41}w_a^4\sin(4m\pi\overline{x})\sin(2n\pi\overline{y})+\\&w_{42}w_a^4\sin(2m\pi\overline{x})\sin(4n\pi\overline{y})+\\&w_{43}w_a^4\sin(2m\pi\overline{x})\sin(2n\pi\overline{y})\\\overline{\Phi}_4=\;&\varphi_{41}w_a^4\cos(4m\pi\overline{x})\cos(2n\pi\overline{y})+\\&\varphi_{42}w_a^4\cos(2m\pi\overline{x})\cos(4n\pi\overline{y})+\\&\varphi_{43}w_a^4\cos(2m\pi\overline{x})\cos(2n\pi\overline{y})+\\&\varphi_{44}w_a^4\cos(4m\pi\overline{x})+\varphi_{45}w_a^4\cos(4n\pi\overline{y})+\\&\varphi_{46}w_a^4\cos(2m\pi\overline{x})+\varphi_{47}w_a^4\cos(2n\pi\overline{y})-N_{4h}w_a^4\overline{y}^{\,2}/2\end{aligned}\tag{28}$$

至此,已经可以写出大挠度渐近解

$$\begin{aligned}\overline{w}=\;&w_0[w_a\sin(m\pi\overline{x})\sin(n\pi\overline{y})]+\\&w_0^3\begin{bmatrix}w_{31}w_a^3\sin(3m\pi\overline{x})\sin(n\pi\overline{y})+\\w_{32}w_a^3\sin(m\pi\overline{x})\sin(3n\pi\overline{y})\end{bmatrix}+\\&w_0^4\begin{bmatrix}w_{41}w_a^4(\sin4m\pi\overline{x})\sin(2n\pi\overline{y})+\\w_{42}w_a^4\sin(2m\pi\overline{x})\sin(4n\pi\overline{y})+\\w_{43}w_a^4\sin(2m\pi\overline{x})\sin(2n\pi\overline{y})\end{bmatrix}+O(w_0^5),\\\overline{\Phi}=\;&-\overline{N}_{\mathrm{cr}}\overline{y}^2/2+w_0[\varphi_aw_a\cos(m\pi\overline{x})\cos(n\pi\overline{y})]+\\&w_0^2\begin{bmatrix}\varphi_{21}w_a^2\cos(2m\pi\overline{x})+\varphi_{22}w_a^2\cos(2n\pi\overline{y})-\\\overline{N}_{2h}w_a^2\overline{y}^2/2\end{bmatrix}+\\&w_0^3\begin{bmatrix}\varphi_{31}w_a^3\cos(3m\pi\overline{x})\cos(n\pi\overline{y})+\\\varphi_{32}w_a^3\cos(m\pi\overline{x})\cos(3n\pi\overline{y})\end{bmatrix}+\\&w_0^4[\varphi_{41}w_a^4\cos(4m\pi\overline{x})\cos(2n\pi\overline{y})+\\&\varphi_{42}w_a^4\cos(2m\pi\overline{x})\cos(4n\pi\overline{y})+\\&\varphi_{43}w_a^4\cos(2m\pi\overline{x})\cos(2n\pi\overline{y})+\\&\varphi_{44}w_a^4\cos(4m\pi\overline{x})+\varphi_{45}w_a^4\cos(4n\pi\overline{y})+\\&\varphi_{46}w_a^4\cos(2m\pi\overline{x})+\varphi_{47}w_a^4\cos(2n\pi\overline{y})-\\&N_{4h}w_a^4\overline{y}^{\,2}/2]+O(w_0^5)\end{aligned}\tag{29}$$

在式 $\overline{w}$ 中取 $(x,y)=(1/2m,1/2n)$,有

$$w_m=w_aw_0-(w_{31}+w_{32})w_a^3w_0^3\tag{30}$$

式(30)中 w_aw_0 可视为 w_m 的表达式,即

$$w_aw_0=w_m+(w_{31}+w_{32})w_m^3$$

代入式(15)第3式,可得到板的后屈曲的平衡路径

$$\overline{N}_x=\overline{N}_{\mathrm{cr}}+\overline{N}_{2h}w_m{}^2+[\overline{N}_{4h}+2\overline{N}_{2h}(w_{31}+w_{32})]w_m{}^4+\cdots\tag{31}$$

3 计算与结果分析

本研究选取四边简支的 T300/5208 石墨环氧复合材料层合板作为研究对象,层合板的几何参数为:

$a=b=2$ m, $h=0.000\ 138$ m；材料参数为：$E_{22}=9$ GPa, $E_{11}=360$ GPa, $G_{12}=4.5$ GPa, $G_{23}=1.8$ GPa, $G_{13}=G_{12}$, $u_{12}=0.25$, $u_{21}=u_{12}E_{22}/E_{11}$。

为了验证分析的准确性，运用 ABAQUS 有限元软件建立模型并进行分析。在屈曲分析步中，边界条件设置为四边简支，在层合板模型 x 轴向的两边，设置 1 N/m 的压缩载荷，最后输出的特征值即为层合板的屈曲载荷值，线性摄动的最大迭代次数设为 1 000 次，以确保模型计算结果的收敛性。

层合板的后屈曲仿真分析是建立在已求得屈曲载荷和屈曲模态的基础上进行的，采用静态 Riks 法分析层合板的非线性后屈曲问题。与线性摄动—屈曲分析不同，在静态 Riks 分析步中，层合板必须要具有一定的初始缺陷，这样才能保证层合板在轴向压缩荷载的作用下可以出现后屈曲挠度即面外变形。在已有的有限元屈曲分析的结果基础上，将屈曲模态的一定倍数作为初始的缺陷赋予到后屈曲有限元计算模型，然后进行加载分析。同时，在 Riks 分析步进行计算时，必须考虑几何非线性选项，且需要设置最大载荷比例系数作为终止条件。在进行计算中，总迭代数设置为 1 000 次，以确保获得较为完整和明显的后屈曲路径曲线。在层合板模型 x 方向两边施加线性压缩载荷，数值设置为已求得的一阶屈曲载荷值，进而就可以将输出的板的挠度值和载荷比例因子（LPF）进行整理，绘制在不同情况下层合板的后屈曲路径曲线。

3.1　有限元验证分析

图 2 给出了 $N=6, \theta=45°$ 与 $N=4, \theta=30°$ 时后屈曲阶段载荷-挠度曲线的理论解与有限元解的比较。从图 2 可以看出，理论分析与有限元分析预测出的层合板的后屈曲路径是一致的，两者都表达了以轴向压缩荷载为特征的后屈曲强化特性，也即在层合板达到屈曲载荷后不会立刻发生失稳，而是随着面外挠度变形 w 的增加，外载荷也在以某个幅度增加。两种方法之间存在着差值，这是由于理论计算的是无初始缺陷的平面板，而有限元模拟分析的是添加了初始缺陷因子的层合板，这样的差值不可避免。

图 3 为有限元仿真和理论分析计算得到的层合板后屈曲变形模态图，图中算例选用：$N=4, \theta=45°$。图 3(a) 可以看出，层合板在达到屈曲后，将发生面外挠度变形，呈现的是整体一致的后屈曲变形模态。图 3(b) 中理论计算并绘制的屈曲变形模态结果与图 3(a) 也相同，再次验证了本研究理论分析的准确性。

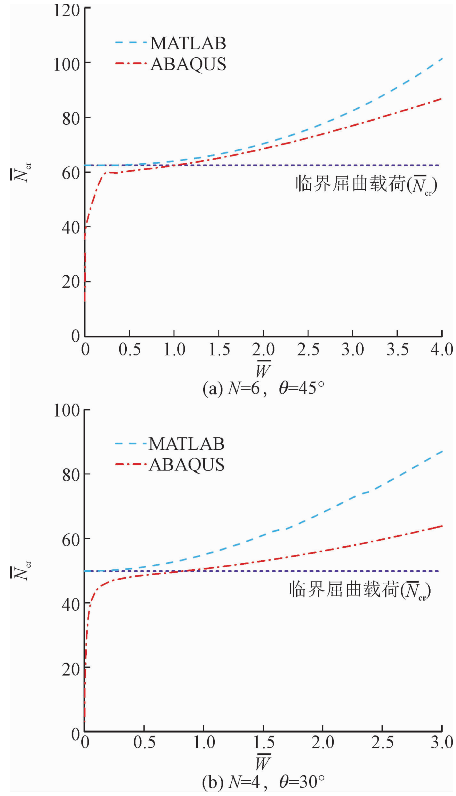

图 2　有限元结果和理论解得到的后屈曲曲线

Fig. 2　Post-buckling curves obtained from finite element results and theoretical solutions

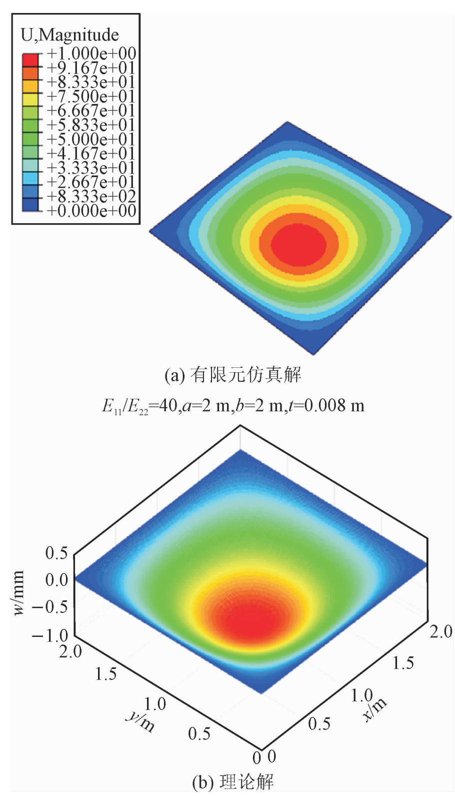

图 3　有限元结果和理论解得到的屈曲变形模态

Fig. 3　Post-buckling deformation modes of finite element results and theoretical solutions

3.2 算例分析

3.2.1 铺设角度对屈曲载荷的影响

通过有限元软件建立了复合材料层合板的模型并进行有限元分析来验证结果的准确性。图4为铺设层数 $N=4,6,8$ 时无量纲屈曲荷载随铺设角度变化的理论解与有限元结果的比较。可以看出,当铺设层数保持不变时,理论解与有限元模拟解趋势一致。当铺设角度增大时,临界屈曲荷载先增大后减小,在 $\theta=45°$ 时到达最大临界屈曲载荷。由公式(21)可知,临界屈曲载荷主要由抗弯刚度决定,相较于其他铺设角度,当 $\theta=45°$ 时,层合板各个方向的抗弯刚度 D_{ij} 整体达到最大,使得临界屈曲载荷最大。

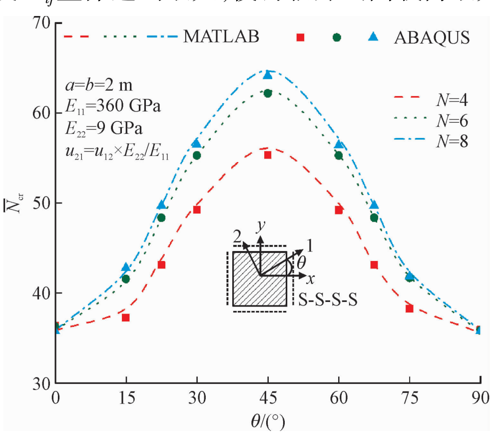

图4 不同铺设层数,临界屈曲载荷随铺设角度的变化

Fig. 4 Effect of fiber orientation angle on buckling load for different number of the laminate

3.2.2 铺设层数对屈曲载荷的影响

图5显示了当铺设角度 $\theta=15°,45°,67.5°$ 时,无量纲屈曲荷载随铺设层数的变化而变化。

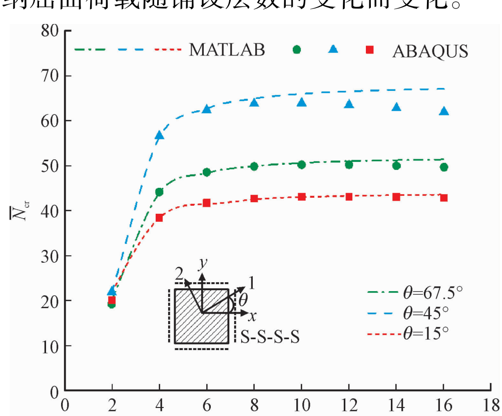

图5 不同铺设角度,临界屈曲载荷随铺设层数的变化

Fig. 5 Effect of the number of the laminate on buckling load for different fiber orientation angle

当铺设角保持度恒定时,随着铺设层数增加,层合板的抗弯刚度增大,临界屈曲载荷呈指数增长。但由于图3纵坐标为无量纲的屈曲载荷,这使得当铺设层数到达一定大小时,对屈曲载荷的增大不再明显,可以看出有限元分析结果与理论结果吻合,两者误差不超过3.32%,验证了理论分析的合理性。

3.2.3 拉弯耦合效应对后屈曲承载能力的影响

为了研究拉弯耦合效应的影响,本研究分析了当铺设方式分别为[45°/45°]对称铺设和[45°/−45°]反对称铺设时,正方形层合板的后屈曲平衡路径。图6给出了两种铺设情况下的方形板的后屈曲路径。图6结果表明,拉弯耦合效应使板的屈后强度大幅度降低。

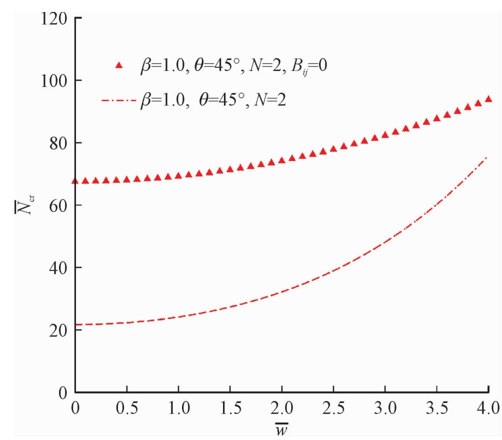

图6 拉弯耦合效应对后屈曲平衡路径的影响

Fig. 6 Influence of coupling effect of tensile on post-buckling equilibrium path

3.2.4 铺设角度对后屈曲承载能力的影响

图7给出了当铺设层数为 $N=4$ 时,不同铺设角度的复合材料层合板的后屈曲平衡路径(四阶近似)。

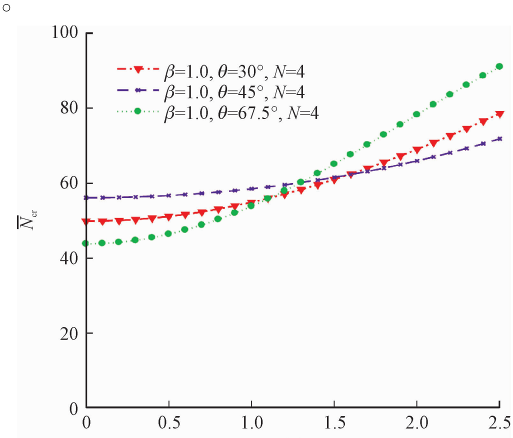

图7 铺设角度对后屈曲平衡路径的影响

Fig. 7 Influence of fiber orientation angle on post-buckling equilibrium path

由图7可知,与 $\theta=30°$ 和 $\theta=67.5°$ 相比,当铺

设角 θ 为 45°时，层合板具有较高的临界屈曲载荷，但后屈曲平衡路径相对更加平缓。显见，在临界屈曲载荷最大的铺设角度时反而对后屈曲的承载能力相对较低，对应不同铺设角时，层合板后屈曲载荷—挠度曲线具有显著差别。

3.2.5　铺设层数对后屈曲承载能力的影响

图 8 显示了在铺设角度 $\theta=45°$时，铺设层数对后屈曲平衡路径的影响。可以看出当铺层数 $N=2$，4，6，8 时，铺层数增加，导致层合板刚度增加，因此初始屈曲载荷也增加，而随着铺层数的增加，拉弯耦合效应减弱，因而后屈曲平衡路径逐趋平缓。

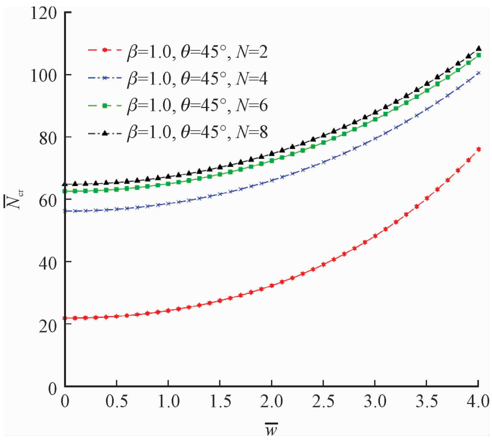

图 8　铺设层数对后屈曲平衡路径的影响

Fig . 8　Influence of the number of the laminate on post-buckling equilibrium path

4　结　论

本研究运用复合材料层合板理论，考虑拉弯耦合效应，研究了反对称铺设层合板的非线性后屈曲问题，通过引入 Von Kármán 非线性几何关系，建立了层合板的非线性后屈曲控制方程，并应用二次摄动法进行求解，获得了求解反对称铺设层合板的临界屈曲荷载及后屈曲平衡路径。开发 MATLAB 软件进行参数分析，详细讨论了不同铺设情况，拉弯耦合效应等对层合板的屈曲荷载与后屈曲平衡路径的影响，同时将理论结果与有限元结果进行对比，验证了本研究分析的准确性与合理性。结论如下。

1）当铺设角度增大时，临界屈曲荷载先增大后减小，在 $\theta=45$ °时到达最大临界屈曲载荷。铺设角的变化将极大地影响板的屈后强度，45°为最佳铺设角。

2）随着铺设层数的增大，层合板的临界屈曲载荷也相应地增加，但后屈曲强化趋势逐趋平缓。

3）拉弯耦合效应使板的屈后强度大大降低，增加铺层数可减弱拉弯耦合效应的影响。

4）与 $\theta=30°$和 $\theta=67.5°$相比，铺设角 θ 为 45°时，层合板具有较高的临界屈曲载荷，但后屈曲平衡路径相对更加平缓，屈后强度比其他铺设角度的强度更低。

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附录 A

三阶摄动方程各系数解分别为

$$y_1=\overline{D}_{11}\alpha^4m^4,y_2=\overline{D}_{22}\beta^4n^4,y_3=(2\overline{D}_{12}+4\overline{D}_{66})\alpha^2\beta^2m^2n^2,y_4=(2\overline{B}_{62}-\overline{B}_{16})\alpha^3\beta m^3n,$$
$$y_5=(2\overline{B}_{61}-\overline{B}_{26})\alpha\beta^3mn^3,$$

$$w_{31}=\frac{2\alpha^2\beta^2m^2n^2\varphi_{21}}{81y_1+y_2+9y_3+27c_{31}y_4+3c_{31}y_5+\dfrac{9\alpha^2\beta^2\overline{N}_{cr}m^2}{\pi^2}},w_{32}=\frac{2\alpha^2\beta^2m^2n^2\varphi_{22}}{y_1+81y_2+9y_3+3c_{32}y_4+27c_{32}y_5+\dfrac{\alpha^2\beta^2\overline{N}_{cr}m^2}{\pi^2}},$$

$$\varphi_{31}=c_{31}w_{31},\varphi_{32}=c_{32}w_{32},\quad N_{2h}=\frac{\pi^2\beta^2n^2}{16\overline{A}_{22}^{-1}\alpha^2m^2}+\frac{\pi^2\alpha^2m^2}{16\overline{A}_{11}^{-1}\beta^2n^2}$$

四阶摄动方程各系数解分别为

$$\varphi_{41}=\frac{c_{411}w_{41}-c_{412}}{c_{413}},\varphi_{42}=\frac{c_{421}w_{42}-c_{422}}{c_{423}},\varphi_{43}=\frac{c_{431}w_{43}-c_{432}}{c_{433}},\varphi_{44}=\frac{5\beta^2n^2w_{31}}{512\alpha^2m^2\overline{A}_{22}^{-1}},\varphi_{45}=\frac{5\alpha^2m^2w_{32}}{512\beta^2n^2\overline{A}_{11}^{-1}},$$
$$\varphi_{46}=-\frac{5\beta^2n^2w_{31}}{32\alpha^2m^2\overline{A}_{22}^{-1}},\varphi_{47}=-\frac{5\alpha^2m^2w_{32}}{32\beta^2n^2\overline{A}_{11}^{-1}}$$

$$w_{41}=\frac{d_{412}+\dfrac{c_{412}d_{411}}{c_{413}}}{d_{413}+\dfrac{c_{411}d_{411}}{c_{413}}},w_{42}=\frac{d_{422}+\dfrac{c_{422}d_{421}}{c_{423}}}{d_{423}+\dfrac{c_{421}d_{421}}{c_{423}}},w_{43}=\frac{d_{432}+\dfrac{c_{432}d_{431}}{c_{433}}}{d_{433}+\dfrac{c_{431}d_{431}}{c_{433}}},$$

$$N_{4h}=\frac{\pi^2(2\overline{B}_{62}-\overline{B}_{16})\alpha mn\varphi_{58}}{\beta}+\frac{\pi^2(2\overline{B}_{61}-\overline{B}_{26})\beta n^3\varphi_{58}}{\alpha m}-2\pi^2n^2(\varphi_a w_{43}+\varphi_{43}+\varphi_{22}w_{32}+\varphi_{21}w_{31}-\varphi_{47}-\varphi_{46})$$

其中，

$$c_{31}=\frac{3(2\overline{\overline{B}}_{16}-\overline{\overline{B}}_{62})\alpha\beta^3mn^3+27(2\overline{\overline{B}}_{26}-\overline{\overline{B}}_{61})\alpha^3\beta m^3n}{81\overline{A}_{22}^{-1}\alpha^4m^4+\overline{A}_{11}^{-1}\beta^4n^4+9(2\overline{A}_{12}^{-1}+\overline{A}_{66}^{-1})\alpha^2\beta^2m^2n^2},$$

$$c_{32}=\frac{27(2\overline{\overline{B}}_{16}-\overline{\overline{B}}_{62})\alpha\beta^3mn^3+3(2\overline{\overline{B}}_{26}-\overline{\overline{B}}_{61})\alpha^3\beta m^3n}{\overline{A}_{22}^{-1}\alpha^4m^4+81\overline{A}_{11}^{-1}\beta^4n^4+9(2\overline{A}_{12}^{-1}+\overline{A}_{66}^{-1})\alpha^2\beta^2m^2n^2},$$

$$c_{411}=32(2\overline{\overline{B}}_{16}-\overline{\overline{B}}_{62})\alpha\beta^3mn^3+128(2\overline{\overline{B}}_{26}-\overline{\overline{B}}_{61})\alpha^3\beta m^3n,$$

$$c_{412}=\frac{5}{2}\alpha^2\beta^2m^2n^2w_{31},$$

$$c_{413}=256\overline{A}_{22}^{-1}\alpha^4m^4+16\overline{A}_{11}^{-1}\beta^4n^4+64(2\overline{A}_{12}^{-1}+\overline{A}_{66}^{-1})\alpha^2\beta^2m^2n^2,$$

$$c_{421}=128(2\overline{\overline{B}}_{16}-\overline{\overline{B}}_{62})\alpha\beta^3mn^3+32(2\overline{\overline{B}}_{26}-\overline{\overline{B}}_{61})\alpha^3\beta m^3n,$$

$$c_{422}=\frac{5}{2}\alpha^2\beta^2m^2n^2w_{32},$$

$$c_{423}=16\overline{A}_{22}^{-1}\alpha^4m^4+256\overline{A}_{11}^{-1}\beta^4n^4+64(2\overline{A}_{12}^{-1}+\overline{A}_{66}^{-1})\alpha^2\beta^2m^2n^2,$$

$$c_{431}=16(2\overline{\overline{B}}_{16}-\overline{\overline{B}}_{62})\alpha\beta^3mn^3+16(2\overline{\overline{B}}_{26}-\overline{\overline{B}}_{61})\alpha^3\beta m^3n,$$

$$c_{432}=\frac{5}{2}\alpha^2\beta^2m^2n^2(w_{31}+w_{32}),$$

$$c_{433}=16\overline{A}_{22}^{-1}\alpha^4m^4+16\overline{A}_{11}^{-1}\beta^4n^4+16(2\overline{A}_{12}^{-1}+\overline{A}_{66}^{-1})\alpha^2\beta^2m^2n^2,$$

$$d_{411}=128(2\overline{B}_{62}-\overline{B}_{16})\alpha^3\beta m^3n+32(2\overline{B}_{61}-\overline{B}_{26})\alpha\beta^3mn^3,$$

$$d_{412}=\alpha^2\beta^2m^2n^2(\varphi_a w_{31}+\varphi_{31}),$$

$$d_{413}=256\overline{D}_{11}\alpha^4m^4+16\overline{D}_{22}\beta^4n^4+64(2\overline{D}_{12}+4\overline{D}_{66})\alpha^2\beta^2m^2n^2-\frac{16\alpha^2\beta^2\overline{N}_{cr}}{\pi^2}m^2,$$

$$d_{421}=32(2\overline{B}_{62}-\overline{B}_{16})\alpha^3\beta m^3n+128(2\overline{B}_{61}-\overline{B}_{26})\alpha\beta^3mn^3,$$

$$d_{422}=\alpha^2\beta^2m^2n^2(\varphi_a w_{32}+\varphi_{32}),$$

$$d_{423}=16\overline{D}_{11}\alpha^4m^4+256\overline{D}_{22}\beta^4n^4+64(2\overline{D}_{12}+4\overline{D}_{66})\alpha^2\beta^2m^2n^2-\frac{4\alpha^2\beta^2\overline{N}_{cr}m^2}{\pi^2},$$

$$d_{431}=16(2\overline{B}_{62}-\overline{B}_{16})\alpha^3\beta m^3n+16(2\overline{B}_{61}-\overline{B}_{26})\alpha\beta^3mn^3,$$

$$d_{432}=\alpha^2\beta^2m^2n^2(4\varphi_a w_{31}-4\varphi_{31}+4\varphi_a w_{32}-4\varphi_{32}),$$

$$d_{433}=16\overline{D}_{11}\alpha^4m^4+16\overline{D}_{22}\beta^4n^4+16(2\overline{D}_{12}+4\overline{D}_{66})\alpha^2\beta^2m^2n^2-\frac{4\alpha^2\beta^2\overline{N}_{cr}m^2}{\pi^2},$$

$$\varphi_{58}=\frac{-2\alpha^2\beta^2m^2n^2w_{43}}{\overline{A}_{22}^{-1}\alpha^4m^4+\overline{A}_{11}^{-1}\beta^4n^4+(2\overline{A}_{12}^{-1}+\overline{A}_{66}^{-1})\alpha^2\beta^2m^2n^2}$$

（编辑　张璐）